Europ. J. Combinatorics (1980) **1**, 9–17

On the Number of Latin Rectangles and Chromatic Polynomial of $L(K_{r,s})$

K. B. ATHREYA, C. R. PRANESACHAR* AND N. M. SINGHI

Using Möbius inversion formula it is shown that the total number of Latin rectangles of a given order can be expressed in terms of Möbius function for the lattice of partitions of a set and the number of colourings of certain graphs. We prove the result in a very general form. In fact, we generalize the notions of Latin rectangles and colourings of graphs and prove a theorem in this general setting. An equivalent form of the theorem which is handy for calculation is given. Various special cases are considered. In particular, we obtain the chromatic polynomials of the line graphs of $K_{3,k}$ and $K_{4,k}$ or equivalently the total number of $3\times k$ and $4\times k$ Latin rectangles with entries from an n-set.

1. INTRODUCTION

Let X be a finite set of n elements. A Latin rectangle [13, p. 35] based on the n-set X is an r by s matrix

$$A=(a_{ij}),\qquad 1\leqslant i\leqslant r,\quad 1\leqslant j\leqslant s,\quad (r,s\leqslant n)$$

such that $a_{ij}\in X$ for all i,j and elements appearing in any row or column of A are distinct.

Let $L(r,s,n)$ denote the total number of Latin rectangles of order $r\times s$ based on an n-set. When $s=n$ we will denote the numbers $L(r,s,n)$ just by $L(r,n)$. Not much is known about $L(r,n)$ or $L(r,s,n)$ for large r, s. In fact only $L(2,n)$ and $L(3,n)$ are known [11, 13] (see also Section 4). For an interesting formula giving $L(3,n)$ in terms of *rencontres* numbers D_n and *ménages* numbers U_n see Riordan [8, 9]. Yamamoto was perhaps the first to give a simple explicit formula for $L(3,n)$. Asymptotic formulas for $L(r,n)$'s have been given by Erdös and Kaplansky [3], Yamamoto [16] and Stein [15]. For a discussion of $L(n,n)$, the number of Latin squares, see [1].

In this paper, we discuss the problem of finding a formula for $L(r,s,n)$. We state the problem in much more generality and derive a formula for this general problem. When specialized, this formula expresses $L(r,s,n)$ as a linear combination of chromatic polynomials of certain graphs obtained from partitions.

Definitions and statements of various results are given in Section 2. A proof of the main theorem is given in Section 3. Our essential tool is Möbius inversion. Section 4 is devoted to applications. We derive some known results and obtain formulas for $L(3,k,n)$ and $L(4,k,n)$, incidentally which also give chromatic polynomials of $L(K_{3,k})$ and $L(K_{4,k})$.

2. STATEMENT OF THE THEOREM

We will denote by J_k the set $\{i\colon 1\leqslant i\leqslant k\}$ of natural numbers. For any finite set A, $|A|$ will denote the cardinality of the set A. Intersections of finite sets will be denoted by juxtaposition. For example, AB will denote $A\cap B$.

For the definition of Möbius function and related results, we refer to Rota [12]. The set of all partitions of a finite set X will be denoted by $\mathbf{P}(X)$. Each member of a partition will

* This author's research is supported by UGC under FIP and partially by TIFR, and sponsored by SDGS College Committee, Hindupur, 515201, India.

0195-6698/80/010009+09$01.00/0

be called a block. Thus a partition P is a set of blocks and $|P|$ denotes the number of blocks in P. The partition whose blocks are singleton sets will be denoted by 0. Given $P, Q \in \mathbf{P}(X)$, we say that P is a refinement of Q and write $P \propto Q$ if each block of P is contained in some block of Q. $\mathbf{P}(X)$ is a partially ordered set ordered by refinement. In fact, it is a lattice. The Möbius function μ for $(\mathbf{P}(X), \propto)$ was calculated by Schützenberger [14] and independently by Frucht and Rota [4]. It is given by

$$\mu(P, Q) = 0 \quad \text{if } P \not\propto Q$$

and $\mu(P, Q) = (-1)^{n_1+n_2+\cdots+n_k-k}(n_1-1)!(n_2-1)!\cdots(n_k-1)!$, if $P \propto Q$, $|Q| = k$ and the k blocks of Q contain $n_1, n_2, \ldots, n_k$ blocks of P.

For the definitions of graphs, chromatic polynomials etc. we refer to Biggs [2]. We will only consider finite, non-directed graphs without multiple edges or loops. We first generalize the notion of vertex-colouring of a graph. Let $G = (X, E)$ be a graph, with X as the vertex-set and E as the edge-set. Let $(A_x\colon x \in X)$ be a family of finite sets indexed by X. By a colouring of G by the family $(A_x\colon x \in X)$, we mean a map $f\colon X \to \bigcup_{x\in X} A_x$, such that $f(x) \in A_x$ and if two vertices x and y are joined in G, then $f(x) \neq f(y)$.

Let $C(G; (A_x\colon x \in X))$ denote the total number of colourings of the graph G by $(A_x\colon x \in X)$. When $A_x = A$ for each $x \in X$, we write this number as $C(G; A)$. Note that if $|A| = \lambda$, then we have $C(G; A) = C(G; \lambda)$, where $C(G; \lambda)$ is the chromatic polynomial of G.

We now define a generalized Latin rectangle. Let $(A_{ij}\colon (i, j) \in J_r \times J_s)$ be a family of finite sets. A generalized Latin rectangle of order $r \times s$ based on the family $(A_{ij}\colon (i, j) \in J_r \times J_s)$ is an $r \times s$ matrix $M = (a_{ij})$, such that $a_{ij} \in A_{ij}$ and any row or column contains distinct elements. The total number of generalized Latin rectangles thus obtained will be denoted by $L(r, s, A_{ij})$ or

$$L\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \cdots & & \cdots & \cdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{pmatrix}.$$

If $A_{ij} = A_j$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, then we will write $L([A_1, A_2, \ldots, A_s]^r)$ for $L(r, s, A_{ij})$ and if $A_{ij} = A$ for all i, j, then we will write $L(r, s, A)$. If $|A| = n$, then $L(r, s, A) = L(r, s, n)$. Note that $L([A_1, A_2, \ldots, A_s]^r)$ is also equal to

$$L\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ A_2 & A_2 & \cdots & A_2 \\ \cdots & & \cdots & \cdots \\ A_s & A_s & \cdots & A_s \end{pmatrix}_{s\times r}.$$

By $L(K_{r,s})$, we will denote the line graph of the complete bipartite graph $K_{r,s}$. The vertex-set for $L(K_{r,s})$ is the set $J_r \times J_s$. Two distinct vertices (i, j) and (i', j') are connected in $L(K_{r,s})$ if either $i = i'$ or $j = j'$. The following obvious result relates chromatic polynomial of $L(K_{r,s})$ and $L(r, s, n)$.

THEOREM 2.1.
(i) $C(L(K_{r,s}); n) = L(r, s, n)$.
(ii) $C(L(K_{r,s}); (A_{ij}\colon (i, j) \in J_r \times J_s)) = L(r, s, A_{ij})$.

Define $R_i = \{(i, j) \in J_r \times J_s\colon 1 \leqslant j \leqslant s\}$, $1 \leqslant i \leqslant r$ and $C_j = \{(i, j) \in J_r \times J_s\colon 1 \leqslant i \leqslant r\}$, $1 \leqslant j \leqslant s$. The sets $R = \{R_1, \ldots, R_r\}$ and $C = \{C_1, \ldots, C_s\}$ will respectively be called the row-partition and column-partition of $J_r \times J_s$.

Define $\mathbf{\Pi}=\{P\in\mathbf{P}(J_r\times J_s): P\propto R\}$, the set of partitions of $J_r\times J_s$ that refine the row partition R. Note that $\mathbf{\Pi}$ can also be considered as $\prod_{i\in J_r}P(R_i)$, and the Möbius function for the partially ordered set $\mathbf{\Pi}$ (ordered by refinement) is same as the product of Möbius functions of the $\mathbf{P}(R_i)$, $1\leq i\leq r$. It is also the restriction to $\mathbf{\Pi}\times\mathbf{\Pi}$ of the Möbius function of $\mathbf{P}(J_r\times J_s)$. (See Rota [12], Propositions 4, 5 of Section 3.)

Given any element $P\in\mathbf{\Pi}$, we can define a graph G_P as follows. Vertices of G_P are the blocks of P. Two distinct vertices A_1 and A_2 are joined if and only if A_1 and A_2 contain elements from the same column, i.e., for some k, $A_1\cap C_k\neq\phi$ and $A_2\cap C_k\neq\phi$.

Given a family of finite sets $\mathcal{A}=(A_{ij}: (i,j)\in J_r\times J_s)$ and $P\in\mathbf{\Pi}$, we will denote by $\mathcal{A}_P$ the family $(A_B: B\in P)$, where $A_B=\bigcap_{(i,j)\in B}A_{ij}$, for each block B of P. Let $\prod_P\mathcal{A}$ denote the cartesian product $\prod_{B\in P}A_B$. We can now state our main theorem.

THEOREM 2.2. *Let $\mathcal{A}=(A_{ij}: (i,j)\in J_r\times J_s)$ be a family of finite sets. Then*

$$L(r,s,\mathcal{A})=\sum_{Q\in\mathbf{\Pi}}\mu(0,Q)C(G_Q;\mathcal{A}_Q). \tag{2.1}$$

COROLLARY 2.3. *$C(L(K_{r,s});n)=L(r,s,n)=\sum_{Q\in\mathbf{\Pi}}\mu(0,Q)C(G_Q;n)$. This corollary follows immediately from the theorem.*

We will also state the theorem in a different form, which is suitable for actual computations. Before that we state a simple result on SDR's. It is an immediate consequence of Möbius inversion (see [5], for a proof). It may be remarked here that Marshall Hall [6] gave a lower bound for the number of Latin rectangles using the method of SDR's.

Given a finite family of finite sets, say $(A_x: x\in X)$, we will denote by $\mathbf{D}(\prod_{x\in X}A_x)$ the set of elements of the cartesian product $\prod_{x\in X}A_x$ with distinct coordinates. In other words, $\mathbf{D}(\prod_{x\in X}A_x)$ is the set of all SDR's for the family $(A_x: x\in X)$ (see Ryser [13], Chapter 5).

THEOREM 2.4. *Let $\mathcal{F}=(A_x: x\in X)$ be a finite family of finite sets. Then*

$$\begin{aligned}\left|\mathbf{D}\Big(\prod_{x\in X}A_x\Big)\right| &= \sum_{P\in\mathbf{P}(X)}\mu(0,P)\prod_{C\in P}\left|\bigcap_{x\in C}A_x\right|\\ &= \sum_{P\in\mathbf{P}(X)}\mu(0,P)\left|\prod_P\mathcal{F}\right|.\end{aligned}\tag{2.2}$$

We will need the expression on the right side of (2.2).

What we actually need is the formal sum $\mathbf{D}(\mathcal{F})$ defined by

$$\mathbf{D}(\mathcal{F})=\sum_{P\in\mathbf{P}(X)}\mu(0,P)\prod_P\mathcal{F}. \tag{2.3}$$

It can be thought of as an element of the vector space $V(\mathcal{F})$ of formal sums $\sum_{P\in\mathbf{P}(X)}\alpha_P\prod_P\mathcal{F}$, ($\alpha_P$ real) with the expressions $\prod_P\mathcal{F}$, $P\in\mathbf{P}(X)$ as a basis.

For any family $\mathcal{A}=(A_{ij}: (i,j)\in J_r\times J_s)$ of finite sets, define $\mathcal{A}_i$ to be the family $(A_{ij}: (i,j)\in R_i)$, $1\leq i\leq r$. Let P_i be a partition of $R_i(1\leq i\leq r)$, so that $P=\bigcup_{i\in J_r}P_i$ is an element of the set $\mathbf{\Pi}$. Now we define the "star" product

$$\Big(\prod_{P_1}\mathcal{A}_1\Big)*\Big(\prod_{P_2}\mathcal{A}_2\Big)*\cdots*\Big(\prod_{P_r}\mathcal{A}_r\Big)$$

to be the integer $C(G_P,\mathcal{A}_P)$. If we extend the definition of $*$ by linearity, we get a function

$$*: V(\mathcal{A}_1)\times V(\mathcal{A}_2)\times\cdots\times V(\mathcal{A}_r)\to\mathbb{R},$$

where $\mathbb{R}$ is the set of real numbers.

Image of any element $(D_1, D_2, \ldots, D_r)$ will be denoted by $D_1 * D_2 * \cdots * D_r$. We now state a second version of Theorem 2.2.

THEOREM 2.5. *Let* $\mathscr{A} = (A_{ij}: (i, j) \in J_r \times J_s)$ *be a family of finite sets. Let* $\mathscr{A}_i = (A_{ij}: (i, j) \in R_i)$, $1 \leq i \leq r$. *Then*

$$L(r, s, \mathscr{A}) = \mathbf{D}(\mathscr{A}_1) * \mathbf{D}(\mathscr{A}_2) * \cdots * \mathbf{D}(\mathscr{A}_r).$$

Theorem 2.5 follows from the observation that the expansion (using linearity of $*$) on the right side of the above equation is precisely the right side of Equation (2.1) of Theorem 2.2, and that the partially ordered set $\mathbf{\Pi}$ is essentially same as the partially order set $\mathbf{P}(R_l) \times \cdots \times \mathbf{P}(R_r)$.

This version has the advantage that when $A_{ij} = A_j$, $1 \leq i \leq r$, we can use multinomial theorem for a quicker expansion.

Finally we remark that the role of rows and columns can be interchanged in the above discussion.

3. Proof of Theorem 2.2

The proof is actually not difficult. As is clear from the statement, it is essentially an example of Möbius inversion.

Suppose we are given a family $\mathscr{A} = (A_{ij}: (i, j) \in J_r \times J_s)$ of finite sets. Let Λ be the set of all $r \times s$ matrices $M = (a_{ij})$ such that $a_{ij} \in A_{ij}$ and elements in each column are distinct, i.e. $a_{ij} \neq a_{i'j}$ if $i \neq i'$, for all j. Note that there is no restriction on rows.

Let $M \in \Lambda$, $M = (a_{ij})$. Define an equivalence relation $\sim$ on $J_r \times J_s$ as follows: $(i, j) \sim (i', j')$ if and only if $i = i'$ and $a_{ij} = a_{i'j'}$. The equivalence classes define a partition of $J_r \times J_s$, which clearly is an element of $\mathbf{\Pi}$. We denote this partition by P_M.

Now for any $Q \in \mathbf{\Pi}$ we define two sets,

$$S_Q = \{M \in \Lambda: Q = P_M\},$$

$$T_Q = \{M \in \Lambda: Q \propto P_M\}.$$

If $Q = 0 \in \mathbf{\Pi}$, then $|S_Q|$ is precisely $L(r, s, \mathscr{A})$. We have $T_Q = \bigcup_{Q \propto Q'} S_{Q'}$ and for $Q' \neq Q''$, $S'_Q \cap S''_Q = \phi$.

Hence $|T_Q| = \sum_{Q \propto Q'} |S_{Q'}|$.

Using Möbius inversion we get,

$$|S_q| = \sum_{Q \propto Q'} \mu(Q, Q')|T_{Q'}|,$$

where μ is the Möbius function for the partially ordered set $\mathbf{\Pi}$.

In particular,

$$L(r, s, \mathscr{A}) = |S_0| = \sum_{Q \in \mathbf{\Pi}} \mu(0, Q)|T_Q|.$$

It is easy to see that $|T_Q|$ is $C(G_Q; \mathscr{A}_Q)$ since each matrix in T_Q clearly gives rise to a colouring of the graph G_Q with respect to the family $\mathscr{A}_Q$, where $\mathscr{A} = (A_{ij}: (i, j) \in J_r \times J_s)$ and conversely each such colouring defines a matrix $M \in T_Q$, and this correspondence is one-to-one.

Hence $L(r, s, \mathscr{A}) = \sum_{Q \in \mathbf{\Pi}} \mu(0, Q)C(G_Q; \mathscr{A}_Q)$. This completes the proof.

4. Applications

Before proceeding with the applications of Theorem 2.2, let us try to analyse what exactly the theorem does. To this end, we state a very general problem.

ONE-ONE PROBLEM. *Let $\mathscr{A}=\{A_1,\ldots,A_s\}$ be a family of s finite sets (not necessarily disjoint) and $X=\bigcup_{1\leq i\leq s}A_i$. The problem is to find the total number $M(\mathscr{A},n)$ of maps $f:X\to J_n$ such that $f|A_i$ (restriction of f to A_i) is one-one for each i, $1\leq i\leq s$.*

One-one problem may be thought of as a problem of finding the total number of strong colourings of a hypergraph.

One can easily see that the problem of finding the total number of colourings of a graph or the numbers $L(r,s,n)$ etc. are particular cases of this problem. It is also clear that the number $M(\mathscr{A},n)$ for the family depends only on the cardinalities of Boolean atoms, i.e., the sets

$$A_T=\Big(\bigcap_{i\in T}A_i\Big)\Big\backslash\Big(\bigcup_{i\notin T}A_i\Big),\quad \text{where } T\subset J_s$$

and, of course, on n.

What one can hope is to give a simple expression for the numbers $M(\mathscr{A},n)$ in terms of the cardinalities of the sets A_T, $T\subset J_s$.

We will give shortly such expressions for families $\mathscr{A}$ which have at most four members. These expressions will be used later.

We note that for $X=J_r\times J_s$ and $\mathscr{A}=\{R_1,R_2,\ldots,R_r,C_1,C_2,\ldots,C_s\}$ the number $M(\mathscr{A},n)$ is precisely $L(r,s,n)$, while for each partition $P\in\Pi$, $\mathscr{A}'=\{A_j\colon 1\leq j\leq s\}$, where $A_j=\{B\in P\colon B\cap C_j\neq\phi\}$, the number $M(\mathscr{A},n)$ is precisely $C(G_P,n)$. Thus Theorem 2.2 reduces a one-one problem on a family of $(r+s)$ sets to one-one problems on families of min. $\{r,s\}$ sets.

We consider now the one-one problem for small families. Suppose $\mathscr{A}=\{A_1,\ldots,A_k\}$ is a family of k finite sets. For any set $S\subset J_k$, define

$$\alpha_s=\left|\Big(\bigcap_{i\in s}A_i\Big)\Big\backslash\Big(\bigcup_{i\notin s}A_i\Big)\right|.$$

If $S=\{i_1,\ldots,i_s\}$, then we will also write $\alpha_{i_1,i_2\ldots i_s}$ for α_s. Thus

$$\alpha_{12}=\alpha_{21}=\left|(A_1A_2)\Big\backslash\Big(\bigcup_{i\geq 2}A_i\Big)\right|,$$

$$\alpha_{123}=\alpha_{231}=\cdots=\left|(A_1A_2A_3)\Big\backslash\Big(\bigcup_{i\geq 3}A_i\Big)\right|\ \text{etc.}$$

The following expressions for $M(\mathscr{A},n)$ can be obtained easily.

(a) If $k=2$, $\mathscr{A}=\{A_1,A_2\}$, then

$$M(\mathscr{A},n)=\begin{bmatrix}n\\ \alpha_{12}\end{bmatrix}\begin{bmatrix}n-\alpha_{12}\\ \alpha_1\end{bmatrix}\begin{bmatrix}n-\alpha_{12}\\ \alpha_2\end{bmatrix},\tag{4.1}$$

where $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]=x!/(x-y)!$ if $x\geq y$ and 0 otherwise. We will denote this number by $\Phi_2(\alpha_1,\alpha_2,\alpha_{12},n)$.

(b) If $k=3$, $\mathscr{A}=\{A_1,A_2,A_3\}$, then

$$M(\mathscr{A},n)=\begin{bmatrix}n\\ \alpha_{12}+\alpha_{13}+\alpha_{23}+\alpha_{123}\end{bmatrix}\begin{bmatrix}n-\alpha_{12}-\alpha_{13}-\alpha_{123}\\ \alpha_1\end{bmatrix}\begin{bmatrix}n-\alpha_{12}-\alpha_{23}-\alpha_{123}\\ \alpha_2\end{bmatrix}$$
$$\times\begin{bmatrix}n-\alpha_{13}-\alpha_{23}-\alpha_{123}\\ \alpha_3\end{bmatrix}.\tag{4.2}$$

This number will be denoted by $\Phi_3(\alpha_1,\alpha_2,\alpha_3,\alpha_{12},\alpha_{13},\alpha_{23},\alpha_{123},n)$ or by $\Phi_3(\{\alpha_3\}_{\substack{s\subset J_3\\ s\neq\phi}},n)$.

(c) If $k=4$, $\mathscr{A}=\{A_1, A_2, A_3, A_4\}$, then

$$M(\mathscr{A}, n) = n!\,V \prod_{i=1}^{4} \frac{(n-|A_i|+\alpha_i)!}{(n-|A_i|)!}, \tag{4.3}$$

where

$$V = \sum_{\theta_1,\theta_2,\theta_3\geqslant 0} \frac{T_{\theta_1\theta_2\theta_3}}{(n_0+\theta_1+\theta_2+\theta_3)!}, \quad n_0 = n - \sum_{\alpha_s},$$

the summation extending over all $S \subset J_4$, $|S|>1$,

$$T_{\theta_1\theta_2\theta_3} = \binom{\alpha_{12}}{\theta_1}\binom{\alpha_{34}}{\theta_1}\theta_1!\binom{\alpha_{13}}{\theta_2}\binom{\alpha_{24}}{\theta_2}\theta_2!\binom{\alpha_{14}}{\theta_3}\binom{\alpha_{23}}{\theta_3}\theta_3!.$$

We will denote this number by $\Phi_4(\alpha_1, \alpha_2, \ldots, \alpha_{1234}, n)$ or by $\Phi_4(\{\alpha_s\}_{s\subset J_4,\, s\neq\phi}, n)$.

(The number $T_{\theta_1\theta_2\theta_3}$ appears since θ_1 members of $A_1A_2\backslash(A_3\cup A_4)$ can be paired with θ_1 members of $A_3A_4\backslash(A_1\cup A_2)$ and these pairs can have the same image etc.]

We now obtain formulas for Latin rectangles with at most four rows (or columns).

THEOREM 4.1 (TWO-LINE LATIN RECTANGLES)

(i) *If* $|A_1|=m$, $|A_2|=n$, $|A_1A_2|=p$, *then*

$$L\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ A_2 & A_2 & \cdots & A_2 \end{pmatrix}_{2\times r} = L([A_1, A_2]^r) = \sum_{k\geqslant 0} (-1)^k \binom{r}{k} \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} m-k \\ r-k \end{bmatrix} \begin{bmatrix} n-k \\ r-k \end{bmatrix}.$$

(ii) $$C(L(K_{2,r}); n) = L(2, r, n) = \begin{bmatrix} n \\ r \end{bmatrix} \sum_{k\geqslant 0} (-1)^k \binom{r}{k} \begin{bmatrix} n-k \\ r-k \end{bmatrix}.$$

(iii) $$L(2, n) = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

PROOF. Using Theorem 2.5, we have

$$L([A_1, A_2]^r) = (A_1\times A_2 - A_1A_2)^{(r)},$$

where $f^{(r)}$ means $f * f * \cdots * f$ (r times). Binomial expansion of the right side gives,

$$L([A_1, A_2]^r) = \sum_{k\geqslant 0} (-1)^k \binom{r}{k} ((A_1\times A_2)^{(r-k)} * (A_1A_2)^{(k)}).$$

Now it can be easily seen that $(A_1\times A_2)^{(r-k)} * (A_1A_2)^{(k)}$ is same as the number of functions

$$f: A\cup B \to A_1 \cup A_2$$

where A and B are two sets such that $|A|=|B|=r$ and $|AB|=k$ and f satisfies $f(A)\subset A_1, f(B)\subset A_2$.

Then by using a method similar to the one used to obtain Equation (4.1), it follows that

$$(A_1\times A_2)^{(r-k)} * (A_1A_2)^{(k)} = \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} m-k \\ r-k \end{bmatrix} \begin{bmatrix} n-k \\ r-k \end{bmatrix}.$$

Thus

$$L([A_1, A_2]^r) = \sum_{k\geqslant 0} (-1)^k \binom{r}{k} \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} m-k \\ r-k \end{bmatrix} \begin{bmatrix} n-k \\ r-k \end{bmatrix}.$$

This proves (i). Statement (ii) follows from (i) by taking $A_1 = A_2 = J_n$. Also (iii) follows from (ii) by putting $r = n$.

This completes the proof.

THEOREM 4.2 (THREE-LINE LATIN RECTANGLES)

(i) $L(3, k, n) = C(L(K_{3,k}); n)$

$$= \frac{n!k!}{((n-k)!)^3} \sum_{\alpha+\beta+\gamma=k} (-1)^{\beta} 2^{\gamma} \frac{((n-k+\alpha)!)^2}{\alpha!\gamma!} \binom{3n-3k+3\alpha+\beta+2}{\beta}.$$

(ii)
$$L(3, n) = (n!)^2 \sum_{\alpha+\beta+\gamma=n} (-1)^{\beta} 2^{\gamma} \frac{\alpha!}{\gamma!} \binom{3\alpha+\beta+2}{\beta}.$$

PROOF. Let A, B, C be any three finite sets. Using Theorem 2.5, we have

$$L\begin{pmatrix} A & A & \cdots & A \\ B & B & \cdots & B \\ C & C & \cdots & C \end{pmatrix}_{3\times k} = L([A, B, C]^k)$$

$$= (A \times B \times C - A \times BC - B \times CA - C \times AB + 2ABC)^{(k)}$$

$$= \sum_{\alpha+\beta_1+\beta_2+\beta_3+\gamma=k} (-1)^{\beta_1+\beta_2+\beta_3} \frac{k!}{\alpha!\beta_1!\beta_2!\beta_3!\gamma!} 2^{\gamma} C_{\alpha\beta_1\beta_2\beta_3\gamma},$$

where

$$C_{\alpha\beta_1\beta_2\beta_3\gamma} = (A \times B \times C)^{(\alpha)} * (A \times BC)^{(\beta_1)} * (B \times CA)^{(\beta_2)} * (C \times AB)^{(\beta_3)} * (ABC)^{(\gamma)}.$$

Taking $A = B = C = J_n$ and using Equation (4.2) above it can be shown that

$$C_{\alpha\beta_1\beta_2\beta_3\gamma} = \Phi_3(\alpha+\beta_1, \alpha+\beta_2, \alpha+\beta_3, \beta_1, \beta_2, \beta_3, \gamma, n).$$

Hence

$$L(3, k, n) = \sum_{\alpha+\beta_1+\beta_2+\beta_3+\gamma=k} (-1)^{\beta_1+\beta_2+\beta_3} \frac{2^{\gamma}k!}{\alpha!\beta_1!\beta_2!\beta_3!\gamma!} \Phi_3(\alpha+\beta_1, \ldots, n)$$

$$= \sum_{\alpha+\beta_1+\beta_2+\beta_3+\gamma=k} (-1)^{\beta_1+\beta_2+\beta_3} \frac{2^{\gamma}k!}{\alpha!\beta_1!\beta_2!\beta_3!\gamma!} \frac{n!}{(n-\beta_1-\beta_2-\beta_3-\gamma)!}$$
$$\times \frac{(n-\beta_2-\beta_3-\gamma)!(n-\beta_3-\beta_1-\gamma)!(n-\beta_1-\beta_2-\gamma)!}{((n-k)!)^3}$$

$$= \frac{n!k!}{((n-k)!)^3} \sum_{\alpha+\beta+\gamma=k} (-1)^{\beta} 2^{\gamma} \frac{((n-k+\alpha)!)^2}{\alpha!\gamma!}$$
$$\times \left\{ \sum_{\beta_1+\beta_2+\beta_3=\beta} \binom{n-k+\alpha+\beta_1}{\beta_1} \binom{n-k+\alpha+\beta_2}{\beta_2} \binom{n-k+\alpha+\beta_3}{\beta_3} \right\}$$

$$= \frac{n!k!}{((n-k)!)^3} \sum_{\alpha+\beta+\gamma=k} (-1)^{\beta} 2^{\gamma} \frac{((n-k+\alpha)!)^2}{\alpha!\gamma!} \binom{3n-3k+3\alpha+\beta+2}{\beta}.$$

This proves (i). By taking $k = n$ in (i) we get (ii). This completes the proof.

We remark that Theorem 4.1 is a well-known result. It appears that statement (ii) of Theorem 4.2 was first obtained by Yamamoto (we do not have access to his paper). Riordan [10] used it to get a recurrence formula for $L(3, n)$. Incidentally, Kerawala [7] was the first to give a "pure" recurrence relation for $L(3, n)$.

Suppose A, B, C are three sets such that $A \cup B \cup C = A \triangle B \triangle C$, where $\triangle$ denotes symmetric difference. Further, let $|A| = l$, $|B| = m$, $|C| = n$ and $|ABC| = s$. Then by methods similar to those used to prove Theorem 4.2, it can be shown that $L([A, B, C]^k)$ is equal to

$$\frac{k!s!}{(l-k)!(m-k)!(n-k)!} \sum_{\alpha+\beta+\gamma=k} (-1)^{\beta} 2^{\gamma} \frac{(l-k+\alpha)!(m-k+\alpha)!(n-k+\alpha)!}{(s-k+\alpha)!\alpha!\gamma!} \times \binom{l+m+n-3k+3\alpha+\beta+2}{\beta}.$$

We also note that we can obtain a formula for $L([A, B, C]^k)$ by these methods for any sets A, B, C. However, the formula involves too many terms.

THEOREM 4.3 (FOUR-LINE LATIN RECTANGLES)

$$L(4, k, n) = C(L(K_{4,k}); n) = \frac{n!k!}{((n-k)!)^4} \sum \frac{(-1)^{\Sigma\beta_i+\varepsilon} 2^{\Sigma\delta_i} 6^{\varepsilon}}{\alpha! \prod(\beta_i!) \prod(\gamma_i!) \prod(\delta_i!) \varepsilon!} \times T \times S,$$

where the sum is over all

$$\alpha + \sum_{i=1}^{6} \beta_i + \sum_{i=1}^{3} \gamma_i + \sum_{i=1}^{4} \delta_i + \varepsilon = k.$$

Further $\prod(\beta_i!) = \prod_{i=1}^{6} \beta_i!$ *etc.*,

$$T = \sum_{\theta_1, \theta_2, \theta_3 \geq 0} \binom{\beta_1+\gamma_1}{\theta_1}\binom{\beta_6+\gamma_1}{\theta_1}\theta_1!\binom{\beta_2+\gamma_2}{\theta_2}\binom{\beta_5+\gamma_2}{\theta_2}\theta_2! \times \binom{\beta_3+\gamma_3}{\theta_3}\binom{\beta_4+\gamma_3}{\theta_3}\theta_3!(n-(\Sigma\beta_i+2\Sigma\gamma_i+\Sigma\delta_i+\varepsilon)+\theta_1+\theta_2+\theta_3)!.$$

and

$$S = (n-k+\alpha+\beta_4+\beta_5+\beta_6+\delta_1)!(n-k+\alpha+\beta_2+\beta_3+\beta_6+\delta_2)! \times(n-k+\alpha+\beta_1+\beta_3+\beta_5+\delta_3)!(n-k+\alpha+\beta_1+\beta_2+\beta_4+\delta_4)!.$$

PROOF. The proof is analogous to that of Theorem 4.2. If A, B, C, D are any four finite sets, then by Theorem 2.4, we have

$$L\begin{pmatrix} A & A & \cdots & A \\ B & B & \cdots & B \\ C & C & \cdots & C \\ D & D & \cdots & D \end{pmatrix}_{4\times k} = L([A, B, C, D]^k) = \Big(A\times B\times C\times D - \sum_{6} A\times B\times CD + \sum_{3} AB\times CD + 2\sum_{4} ABC\times D - 6(ABC)\Big)^{(k)},$$

in which the number under each $\sum$ denotes the number of terms associated with it. If $A = B = C = D = J_n$, then we get $L(4, k, n)$. We omit the details.

We only note that we have to use (C) to get the expression stated in the theorem. A satisfactory simplification of this expression seems to be difficult. Finally, $L(4, n)$ is obtained by putting $k = n$ in this expression.

References

1. R. Alter, How many Latin squares are there? *Amer. Math. Monthly* **82** (1975), 632–634.
2. N. L. Biggs, *Algebraic Graph Theory*, Cambridge University Press (1974).
3. P. Erdös and J. Kaplansky, The asymptotic number of Latin rectangles, *Amer. J. Math.* **68** (1946), 230–236.
4. R. Frucht and G.-C. Rota, La funci6n de Möbius para el retículo di particiones de un conjunto finito, Scientia (Chile).
5. J. E. Graver and M. E. Watkins, *Combinatorics with Emphasis on the Theory of Graphs*, Springer-Verlag (GTM 54) (1977).
6. M. Hall, Jr., *Combinatorial Theory*, Blaisdell Publishing Company (1967).
7. S. M. Kerawala, The enumeration of the Latin rectangle of depth three by means of difference equation, *Bull. Calcutta Math. Soc.* **33** (1941), 119–127.
8. J. Riordan, Three-line Latin rectangles, *Amer. Math. Monthly* **51** (1944), 450–452.
9. J. Riordan, Three-line Latin rectangles—II, *Amer. Math. Monthly* **53** (1946), 18–20.
10. J. Riordan, A recurrence relation for three-line Latin rectangles, *Amer. Math. Monthly* **59** (1952), 159–162.
11. J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley and Sons, Inc. (1958).
12. G.-C. Rota, On the foundations of Combinatorial Theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeits theories* **2** (1964), 340–368.
13. H. J. Ryser, *Combinatorial Mathematics*, The Mathematical Association of America (1963).
14. M. P. Schüzenberger, Contribution aux applications statisques de la théorie de l'information, *Publ. Inst. Stat. Univ. Paris* **3** (1954), 5–117.
15. C. M. Stein, Asymptotic evaluation of the number of Latin rectangles, *J. Combinatorial Theory Ser. A* **25** (1978), 38–49.
16. K. Yamamoto, On the asymptotic number of Latin rectangles, Jap. J. Math. **21** (1951), 113–119.

(*Received* 18 *May* 1979)

K. B. Athreya and C. R. Pranesachar
Department of Applied Mathematics, Indian Institute of Science,
Bangalore 560012, India

N. M. Singhi
School of Mathematics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Bombay 400005, India